laminar boundary-layers Part 4: Universal series solutions," Wright Air Dev Center TR 53 288 (1954)

[4] Terrill, R M , "Laminar boundary-layer flow near separation with and without suction," Phil Trans Roy Soc (London) **A253,** 55–100 (1960)

Reply by Author to R M Terrill

Robert L Kosson*
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THE writer had not been aware of Terrill's exact solution[1] for sinusoidal velocity distribution and is pleased at the closeness of agreement of the approximate solution[2] with it

In dealing with the approximate solution near the separation point, it should be noted that a transition from use of the polynomial inner solution [Eqs (33–37)] to use of the von Kármán Millikan inner solution [Eqs (29–32)] takes place when the velocity profile inflection point occurs at a dimensionless stream-function value z greater than 0 15 A small discontinuity in the boundary-layer parameters is associated with this transition, as shown in Table 1 for the sinusoidal velocity distribution Note that the nondimensional shear-stress value reported in Table 4 of Ref 2 for $\eta = 100°$ was in error, and should be 0 40

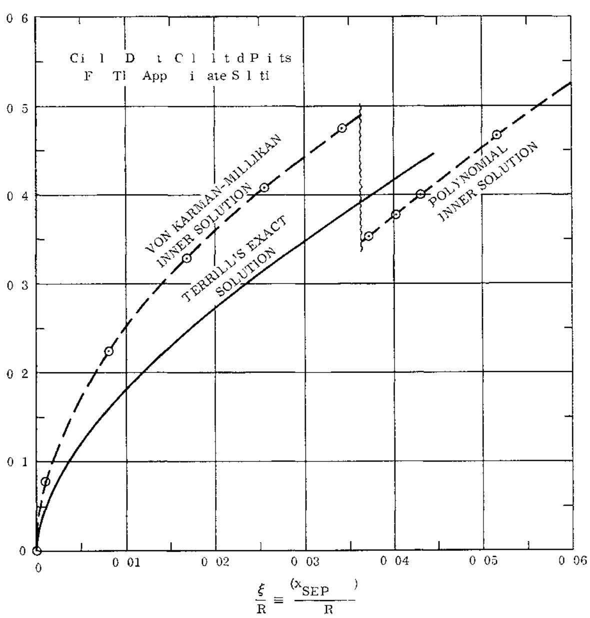

Fig 1 Shear stress near separation point, sinusoidal velocity distribution, $U_a = 2U_\infty \sin(X/R)$

Although the approximate solution errs in predicting separation too soon (in this example), the variation of shear stress with distance from the separation point is similar to that calculated by Terrill, as shown in Fig 1 The good agreement in displacement and momentum thickness at the separation point is related to this

Also, it may be noted that the value of $C = 0\ 788$ used in the approximate solution is, to some extent, arbitrary and was determined at the forward stagnation point It is possible that a different value of C would give better agreement in the vicinity of the separation point

Table 1 Approximate solution results for sinusoidal velocity distribution

η	Inner solution	$\frac{\tau_0}{\rho U_\infty^2}\left(\frac{U_\infty R}{\nu}\right)^{1/2}$	$\delta^*\left(\frac{U_\infty R}{\nu}\right)^{1/2}$	$\theta\left(\frac{U_\infty R}{\nu}\right)^{1/2}$
0	Polynomial	0	0 456	0 203
30	"	1 62	0 481	0 212
60		2 22	0 580	0 250
90		1 26	0 918	0 357
100		0 40	1 372	0 443
100 33		0 35	1 409	0 447
100 5	von Kármán-Millikan	0 47	1 328	0 443
101	"	0 41	1 376	0 449
101 5	"	0 33	1 438	0 456
102		0 22	1 527	0 463
102 40		0 08	1 671	0 471
102.46		0.00	1.759	0 472

References

[1] Terrill, R M , "Laminar boundary-layer flow near separation with and without suction," Phil Trans Roy Soc (London) **A253,** 55–100 (1960)

[2] Kosson, R L , "An approximate solution for laminar boundary layer flow,' AIAA J **1,** 1088–1096 (1963)

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Comment on "A Class of Linear Magnetohydrodynamic Flows"

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IN a recent note[1] the characteristics of steady-state magnetohydrodynamic flows of an incompressible, electrically conducting fluid with constant scalar properties were examined with respect to the functional dependence of the solutions that would be necessary to linearize the governing equations Two cases were specifically considered and solutions presented Closer examination of the problem shows that the results obtained are more restrictive than is immediately obvious, and the following analysis is presented to show exactly the limitations implied by the interactions among the assumptions in each of the two cases The symbols have their usual meaning, and the mks system of units has been used

In case 1, the form of the velocity and magnetic field [$\mathbf{V} = \mathbf{i}u(y,z)$ and $\mathbf{B} = \mathbf{i}B_x(y,z) + \mathbf{k}B_0$] is such that the continuity equation and the divergence relation for $\mathbf{B}$ are identically satisfied An expansion of Ampere's law shows that $J_x = 0$, and a subsequent expansion of Ohm's law shows that $E_x = 0$ Eliminating the current between Ohm's law and the divergence relation, and acknowledging the divergence relation for $\mathbf{E}$ gives the result $\partial u/\partial y = 0$ or, integrating, $u = u(z)$ This shows that the form of the assumed relation for the velocity is too general to be compatible with the functional forms of the other variables

The momentum equation may now be expanded to give

$$\partial P/\partial x = B_0 J_y + \mu(d^2u/dz^2) \tag{1}$$

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$$\partial P/\partial y = B_x J \tag{2}$$

$$\partial P/\partial z = -B_x J_y \tag{3}$$

Taking the partial derivative of Eqs (2) and (3) with respect to x [since $\mathbf{J} = \mathbf{J}(y,z)$] and integrating with respect to y and z, respectively, shows that the pressure gradient in the x direction may be no more than a function of x Further examination of the functional dependences in Eq (1) shows that it must be a constant, or

$$\partial P/\partial x = -C_1 \tag{4}$$

Eliminating the current between Ampere's law and the momentum equation and then integrating the partial derivatives gives

$$P = -C_1 x - [B_x^2 + f_1(z)]/2\mu_0$$

$$P = -C_1 x - [B_x^2 + f_2(y)]/2\mu_0$$

Obviously, the pressure must be

$$P = -C_1 x - (B_x^2 + C_2)/2\mu_0$$

Since the velocity is a function of z only, and the pressure gradient in the x direction is a constant, Eq (1) shows that $J_y = J_y(z)$ Introducing this into the divergence relation and integrating gives $J = J(y)$ Similarly, from Ohm's law and the divergence relation for $\mathbf{E}$ [since $\mathbf{E} = \mathbf{E}(y,z)$] it follows that

$$E = E(y) \tag{5}$$

$$E_y = E_y(z) \tag{6}$$

Faraday's law may now be expanded, and, on the basis of the results of Eqs (5) and (6), it gives

$$E_y = C_3 z + C_4 \tag{7}$$

$$E = C_3 y + C_5 \tag{8}$$

Following from Ohm's law,

$$\begin{aligned} J_y &= C_3\sigma z + C_4\sigma - \sigma B_0 u \\ J &= C_3\sigma y + C_5\sigma \end{aligned} \tag{9}$$

Combining Eqs (1, 4, and 9) gives an ordinary differential equation for the velocity:

$$\frac{d^2u}{dz^2} - \left(\frac{\sigma B_0^2}{\mu}\right) u = -C_3\left(\frac{\sigma B_0}{\mu}\right) z - C_4\left(\frac{\sigma B_0}{\mu}\right) - C_1\left(\frac{1}{\mu}\right)$$

The solution for the velocity is, therefore,

$$u = C_6 \exp\left[\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] + C_7 \exp\left[-\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] + C_3\left(\frac{1}{B_0}\right) z + C_4\left(\frac{1}{B_0}\right) + C_1\left(\frac{1}{\sigma B_0^2}\right) \tag{10}$$

Combining Ampere's law and Ohm's law and introducing the solutions of Eqs (7) and (8) gives

$$\partial B_x/\partial y = -C_3\mu_0\sigma y - C_5\mu_0\sigma$$

$$\partial B_x/\partial z = C_3\mu_0\sigma z + C_4\mu_0\sigma - \mu_0\sigma B_0 u$$

After including the solution for the velocity, Eq (10), these can be integrated to give the solution for the magnetic field:

$$B_x = -C_6(\mu\mu_0^2\sigma)^{1/2} \exp\left[\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] + C_7(\mu\mu_0^2\sigma)^{1/2} \times \exp\left[-\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] - C_3\left(\frac{\mu_0\sigma}{2}\right) y^2 - C_5(\mu_0\sigma) + C_8$$

The solutions just determined show that the correct description of case 1 flows is therefore,

$$\mathbf{V} = \mathbf{i}u(z)$$

$$\mathbf{B} = \mathbf{i}[B_{x1}(y) + B_{x2}(z)] + \mathbf{k}B_0$$

$$P = P_1(x) + P_2(y) + P_3(z)$$

$$\mathbf{J} = \mathbf{j}J_y(z) + \mathbf{k}J(y)$$

$$\mathbf{E} = \mathbf{j}E_y(z) + \mathbf{k}E(y)$$

Again in case 2, the form of the velocity and magnetic field [$\mathbf{V} = \mathbf{i}u(z) + \mathbf{j}v(z)$ and $\mathbf{B} = \mathbf{i}B_x(z) + \mathbf{j}B_y(z) + \mathbf{k}B_0$] is such that the continuity equation and the divergence relation for $\mathbf{B}$ are identically satisfied

An expansion of Ampere's law shows that $J = 0$, which also identically satisfies the divergence relation for $\mathbf{J}$ An expansion of Faraday's law [since $\mathbf{E} = \mathbf{E}(z)$] requires that $dE_x/dz = 0$ Thus, $E_x = K_1$ Similarly, $E_y = K_2$ and, from the divergence relation for $\mathbf{E}$, $E = K_3$

The momentum equation may be expanded to give

$$\partial P/\partial x = B_0 J_y + \mu(d^2u/dz^2) \tag{11}$$

$$\partial P/\partial y = -B_0 J_x + \mu(d^2v/dz^2) \tag{12}$$

$$\partial P/\partial z = J_x B_y - J_y B_x \tag{13}$$

Taking the partial derivative of Eqs (12) and (13) with respect to x [since $\mathbf{J} = \mathbf{J}(z)$] and then integrating with respect to y and z, respectively, shows that the pressure gradient in the x direction may be no more than a function of x Similarly, taking the partial derivative of Eqs (11) and (13) with respect to y and then integrating with respect to x and z, respectively, shows that the pressure gradient in the y direction may be no more than a function of y Further examination of the functional dependences in Eqs (11) and (12) show that they must both be constants, or

$$\partial P/\partial x = -K_4 \tag{14}$$

$$\partial P/\partial y = -K_5 \tag{15}$$

Eliminating the current between Ampere's law and the momentum equation and integrating the partial derivatives gives the pressure as

$$P = -K_4 x - K_5 y - (1/2\mu_0)(B_x^2 + B_y^2) + K_6$$

Eliminating the current between Ohm's law and the momentum equation and introducing Eqs (14) and (15) gives the equations for the velocity

$$\frac{d^2u}{dz^2} - \left(\frac{\sigma B_0^2}{\mu}\right) u = -K_2\left(\frac{\sigma B_0}{\mu}\right) - K_4\left(\frac{1}{\mu}\right)$$

$$\frac{d^2v}{dz^2} - \left(\frac{\sigma B_0^2}{\mu}\right) v = K_1\left(\frac{\sigma B_0}{\mu}\right) - K_5\left(\frac{1}{\mu}\right)$$

The solutions for the velocity are, therefore,

$$u = K_7 \exp\left[\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] + K_8 \exp\left[-\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] + K_2\left(\frac{1}{B_0}\right) + K_4\left(\frac{1}{\sigma B_0^2}\right)$$

$$v = K_9 \exp\left[\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] + K_{10} \exp\left[-\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] - K_1\left(\frac{1}{B_0}\right) + K_5\left(\frac{1}{\sigma B_0^2}\right)$$

The currents can now be determined from Ohm's law as

$$J_x = K_9\sigma B_0 \exp\left[\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] + K_{10}\sigma B_0 \times \exp\left[-\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] + K_5\left(\frac{1}{B_0}\right)$$

$$J_y = -K_7\sigma B_0 \exp\left[\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] - K_8\sigma B_0 \times \exp\left[-\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] - K_4\left(\frac{1}{B_0}\right)$$

and these equations can be integrated in accordance with Ampere's law to give the magnetic field as

$$B_x = -K_7(\mu\mu_0^2\sigma)^{1/2} \exp\left[\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] - K_8(\mu\mu_0^2\sigma)^{1/2} \exp\left[-\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] - K_4\left(\frac{\mu_0}{B_0}\right) z + K_{11}$$

$$B_y = -K_9(\mu\mu_0^2\sigma)^{1/2} \exp\left[\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] - K_{10}(\mu\mu_0^2\sigma)^{1/2} \exp\left[-\left(\frac{\sigma B_0^2}{\mu}\right)^{1/2} z\right] - K_5\left(\frac{\mu_0}{B_0}\right) z + K_{12}$$

The solutions just determined show that the correct description of case 2 flows is, therefore,

$$\begin{aligned} \mathbf{V} &= \mathbf{i}u(z) + \mathbf{j}v(z) \\ \mathbf{B} &= \mathbf{i}B_x(z) + \mathbf{j}B_y(z) + \mathbf{k}B_0 \\ P &= P_1(x) + P_2(y) + P_3(z) \\ \mathbf{J} &= \mathbf{i}J_x(z) + \mathbf{j}J_y(z) \\ \mathbf{E} &= \text{const} \end{aligned}$$

In conclusion, the problem encountered in attempting to generalize magnetohydrodynamic flow problems lies in the fact that the governing equations are a set of coupled equations Thus, any restrictions imposed on one variable have effects on the allowable forms for all other variables, and a careful examination of all of the implicit restrictions is essential As has been previously shown, what appears to be a relatively general solution may in fact, turn out to be a rather specialized result

Reference

[1] El-Saden, M R, 'A class of linear magnetohydrodynamic flows,' AIAA J 1, 236–238 (1963)

Reply by Author to T P Anderson

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THE primary purpose of my note[1] was to point out, in a general way, the manner in which the nonlinear terms in the governing equations for incompressible magnetohydrodynamic flow problems may be identically satisfied The boundary conditions outlined in my note do just that If these boundary conditions entail additional restrictions as Anderson has shown, so be it This is beside the primary purpose of my note

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Solutions to the linearized equations are available in several publications, some of which were referred to in the note

Reference

[1] El Saden, M R, 'A class of linear magnetohydrodynamic flows," AIAA J 1, 236–238 (1963)

Comments on "Numerical Analysis of Unsymmetrical Bending of Shells of Revolution"

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Space Technology Laboratories, Inc, Redondo Beach, Calif

THE paper[1] presented by Budiansky and Radkowski has been of interest to the author in connection with a digital program, similar to that of Ref 1, developed at Space Technology Laboratories for static and dynamic analyses of unsymmetrically loaded shells of revolution However, the conditions to be applied at the pole of a surface of revolution which were given in Ref 1 differ from the ones used in the Space Technology Laboratories program, and it is on this aspect of the paper that the author wishes to comment

In Ref 1, it was stated that a simple-minded way to handle the singularity problem that arises at a pole of a surface of revolution is to choose the boundary $S = 0$, not at the pole, but at a very short distance away, and then to impose boundary conditions at $S = 0$ as

$$\left.\begin{aligned} u_\xi = u_\theta = \phi_\xi = \hat{f}_\xi = 0 \qquad & \text{for } n = 0 \\ t_\xi = \hat{t}_{\xi\theta} = w = m_\xi = 0 \qquad & \text{for } n = 1 \\ u_\xi = u_\theta = w = m_\xi = 0 \qquad & \text{for } n \geqslant 2 \end{aligned}\right\} \tag{1}$$

where n is the Fourier index

The conditions just given for $n = 0$ are correct; however, it will be shown that two of the conditions given for $n = 1$ are not independent, and for this reason an additional condition is required; it will also be shown that, for $n = 2$, the condition $m_\xi = 0$ is incorrect Furthermore, it will be shown that the correct conditions can be applied at the pole so that the point $S = 0$ will be chosen to be at the pole

The conditions to be applied at the pole can be determined by examining the strain-displacement and curvature-displacement relations and the equations of equilibrium for $\rho = 0$ The former relations are given in Ref 1 by Eqs (29–30) and become, after simple substitutions,

$$\left.\begin{aligned} e_\xi &= u_\xi' + \omega_\xi w \\ e_\theta &= (1/\rho)\{nu_\theta + \rho' u_\xi + [1 - (\rho')^2]^{1/2} w\} \\ e_{\xi\theta} &= (-1/2\rho)[\rho' u_\theta + nu_\xi - \rho u_\theta'] \\ k_\xi &= [-w' + \omega_\xi u_\xi]' = \phi_\xi' \\ k_\theta &= (1/\rho)^2[\rho\rho'\phi_\xi + n^2 w + n\rho\omega_\theta u_\theta] \\ k_{\xi\theta} &= (1/2\rho)^2\{-n\rho\phi_\xi + n\rho w' - 2n\rho' w - \\ & \quad 2\rho\rho'\omega_\theta u_\theta + \rho\rho'\omega_\xi u_\theta + \rho^2 w_\theta u'_\theta + \\ & \quad \tfrac{1}{2}(\omega_\theta - \omega_\xi)\rho[nu_\xi + \rho' u_\theta + \rho u'_\theta]\} \end{aligned}\right\} \tag{2}$$

At the pole $r = 0$, the following relations hold:

$$\rho = 0 \qquad \rho' = 1 \qquad \rho'' = 0 \qquad \omega_\xi = \omega_\theta \tag{3}$$

From Eqs (2) and (3) one may see that, at the pole, the expressions for the strains e_θ and $e_{\xi\theta}$ have a first-order zero in the denominator, whereas the expressions for the curvatures k_θ and $k_{\xi\theta}$ have a second order zero in the denominator

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